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Invertibility of Irreducible Matrices

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ABSTRACT

We obtain new sufficient conditions for invertibility of an irreducible complex matrix. Remarks are also given on eigenvalues (and the associated eigenvectors) that lie on the boundary of various spectrum inclusion regions of an irreducible matrix. Our results extend, strengthen, or clarify the recent work of Brualdi, Brualdi and Mellendorff, Farid, Solov'ev, and Zhang and Gu. © Elsevier Science Inc., 1997

1. INTRODUCTION

Unless specified otherwise, all matrices considered in this paper are square and complex. Given an $n \times n$ matrix $A = (a_{ij})$, we denote by R_i (C_i)

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the ith deleted absolute row (column) sum of A; that is,

$$R_i = \sum_{j \neq i} |a_{ij}|, \quad C_i = \sum_{j \neq i} |a_{ji}| \qquad (i = 1, 2, \dots, n).$$

The famous classical Geršgorin theorem says that each eigenvalue of A lies in the union of the n disks

$$D_i = \{z : |z - a_{ii}| \leqslant R_i\} \qquad (i = 1, 2, \dots, n)$$

of the complex plane. Generalizations of this result have been obtained by Brauer, Ostrowski, Taussky, and others and are nicely summarized in [6], [16, Chapter 6], and [7, Section 3.6]. (See also [8, pp. 80–82], [22, pp. 35–36], and [17] for recent commentaries on this topic.) In general, a result about an inclusion region for the spectrum of a matrix amounts to providing a necessary condition for a complex number λ (and, in particular, for 0) to be an eigenvalue of the matrix. By putting the result in its contrapositive form, we obtain an equivalent formulation as a sufficient condition for invertibility of matrices. In Theorem A below we collect some of the well-known sufficient conditions for invertibility of matrices that come from spectrum inclusion regions.

For an $n \times n$ matrix A, by the *digraph* of A, denoted by $\Gamma(A)$, we mean as usual the directed graph with vertex set $\{P_1, \dots, P_n\}$ such that there is an arc from P_r to P_s if and only if $a_{rs} \neq 0$. A matrix A is said to be *weakly irreducible* if each vertex of $\Gamma(A)$ belongs to at least one nontrivial (simple, directed) cycle (i.e. cycle of length at least two).

THEOREM A. *Let $A = (a_{ij})$ be an $n \times n$ complex matrix. For each real number α with $0 \leqslant \alpha \leqslant 1$, each of the following is a sufficient condition for A to be invertible, where for conditions* (c) *and* (d) *we need to add the hypothesis that A is weakly irreducible*:

(a) $|a_{ii}| > R_i^{\alpha} C_i^{1-\alpha}$ *for all* $i = 1, \dots, n$.

(b) $|a_{ii}|\,|a_{jj}| > R_i^{\alpha} C_i^{1-\alpha} R_j^{\alpha} C_j^{1-\alpha}$ *for all pairs* (i, j), $1 \leqslant i < j \leqslant n$.

(c) $|a_{ii}|\,|a_{jj}| > R_i^{\alpha} C_i^{1-\alpha} R_j^{\alpha} C_j^{1-\alpha}$ *for all pairs* (i, j), $1 \leqslant i < j$, *for which the vertices* P_i, P_j *lie on a common cycle of* $\Gamma(A)$.

(d) $\prod_{P_i \in \gamma} |a_{ii}| > \prod_{P_i \in \gamma} R_i^{\alpha} C_i^{1-\alpha}$ *for all nontrivial cycles* γ *of* $\Gamma(A)$.

The sufficient condition (a) of Theorem A is due to Ostrowski. When $\alpha = 1$, it reduces to the Levy-Desplanques theorem, which is equivalent to the Geršgorin theorem and says that every strictly diagonally dominant matrix is invertible. Condition (b) is also due to Ostrowski; when $\alpha = 1$, it reduces to Brauer's condition. To the theorem we have added condition (c), which is suggested by the work of Zhang and Gu [25]. When $\alpha = 1$, that (c) is a sufficient condition follows from [25, Theorem 1]. Condition (d) is due to Brualdi [6, Corollary 2.13].

Some general remarks are in order. Since the spectrum of a matrix is just the union of the spectra of the irreducible diagonal blocks that appear in its Frobenius normal form [7, Theorem 3.2.4], in problems concerning spectrum inclusion regions or invertibility of matrices, the irreducible case is the essential case. Indeed, often a better result can be obtained in the irreducible case. For instance, Taussky [21] strengthened the Levy-Desplanques theorem and obtained the result that if $A = (a_{ij})$ is an $n \times n$ diagonally dominant irreducible matrix for which $|a_{kk}| > R_k$ for at least one k, then A is invertible. So a study of this topic would seem incomplete if the irreducible case is not examined. [To be fair, we would add that in the general case when the inequalities (for the sufficient conditions) are all strict, often one can also obtain a positive lower bound for $|\det A|$. See, for instance, [1, 3, 18, 20]. Also, in practical problems when data are not known or cannot be determined exactly, it is desirable to have sufficient conditions given by strict inequalities.] Hence, it is natural to ask whether in the case when A is irreducible, the sufficient conditions given in Theorem A can be relaxed somehow. This paper is the outcome of our attempt to answer this question. Below we describe briefly what is known about this question, and what we are going to do.

As we have mentioned above, when A is irreducible and $\alpha = 1$, Taussky proved that condition (a) of Theorem A can be relaxed by replacing the strict inequalities all by weak inequalities, but keeping at least one strict inequality. Brualdi [6, Theorem 2.9] proved that when $\alpha = 1$ condition (d) can be relaxed in a similar way. Zhang and Gu [25, Theorem 1] showed that a similar remark also holds for condition (c). In the same paper they pointed out an error in an often quoted assertion of Brauer [4, Theorem 22] about an eigenvalue that lies on the boundary of the union of the well-known ovals of Cassini that includes the spectrum of an irreducible matrix. [It is worth mentioning that in [5, Theorems 35, 36, 37] Brauer also provided three other (with two for the real case) smaller, but more complicated and much less well-known ovals of Cassini as spectrum inclusion regions.] In essence, they showed that when $\alpha = 1$ (and A is irreducible), if the strict inequalities in condition (b) of Theorem A are all replaced by weak inequalities but keeping at least one strict inequality, then the condition is no longer sufficient for the invertibility of A. Recently, Zhang and Yang [26] also proved that for a general α, $0 \leqslant \alpha \leqslant 1$, condition (d) can be relaxed by replacing most of the strict inequalities by weak inequalities but keeping at least one strict inequality.

In this paper, we treat the above question in the more general context when each $R_i^\alpha C_i^{1-\alpha}$ is replaced by τ_i, where $(\tau_i, \dots, \tau_n)$ is an n-tuple of positive real numbers with the property that the union of the disks

$$D_{\tau, i} = \{z : |z - b_{ii}| \leqslant \tau_i\} \qquad (i = 1, \dots, n)$$

contains all eigenvalues of $B = (b_{ij})$ for any $n \times n$ matrix B such that $|b_{ij}| = |a_{ij}|$ for all $i \neq j$. We show that in this context conditions (a), (c), and (d) of Theorem A can be relaxed by replacing most of the strict inequalities by weak inequalities but keeping at least one strict inequality. In case of condition (b) we show that, by ruling out matrices of a special form, it can also be relaxed in a similar way. Thus, by choosing different n-tuples $(\tau_1, \dots, \tau_n)$, we obtain different results.

The treatment of the question of invertibility or of spectrum inclusion regions of matrices in this general context is not unknown. It originated with Fan [13] and was quite popular in the 1960s and 1970s (see [10–12, 15]). Indeed, in [6, Theorems 2.12 and 2.14] Brualdi also proved two results about spectrum inclusion regions of an irreducible matrix in this general context. However, results of this type are less well known, probably because they are not available in books or monographs on matrix theory. In the course of answering our question, we also touch upon equivalent conditions for an n-tuple τ with the said property, and offer a simple unified treatment which is based on the theory of M-matrices and a theorem of Brualdi [6, Theorem 2.9].

We also treat the all-equalities cases of our question; that is; if A is an irreducible matrix and if the strict inequalities in condition (a) [or (b), (c), or (d)] of Theorem A are all replaced by equalities and each $R_i^\alpha C_i^{1-\alpha}$ is replaced by τ_i, where $\tau = (\tau_1, \dots, \tau_n)$ is an n-tuple of positive real numbers with properties described before, then what extra assumptions do we need to add in order to guarantee that A is invertible? By examining known proofs carefully, we note that in the all-equalities cases, if A is irreducible and singular and $x = (x_1, \dots, x_n)^T$ is an eigenvector corresponding to 0, then necessarily all x_i are nonzero, and we have $\mathfrak{M}(A)|x| = 0$, where $|x|$ is the *modulus* (vector) of A, i.e. $(|x_1|, \dots, |x_n|)^T$, and $\mathfrak{M}(A) = (m_{ij})$ is the comparison matrix of A given as follows: m_{ij} equals $|a_{ij}|$ for $i = j$ and equals $-|a_{ij}|$ for $i \neq j$. In such a case we find that A must fulfill a certain readily checkable set of equalities; hence, by requiring at least one of the equalities to be a strict inequality, we obtain a sufficient condition for invertibility of A. It is worthwhile to note that in the literature the well-known sufficient conditions for invertibility of A are all satisfied under the circumstance that $\mathfrak{M}(A)$ is a nonsingular M-matrix (in which case, A is called an *H-matrix*; see [24]). In the all-equalities cases we are dealing with, we touch upon the case when $\mathfrak{M}(A)$ is an irreducible singular M-matrix.

As a special case of the all-equalities cases, we consider also invertibility of an irreducible matrix A that satisfies $|a_{ii}| = R_i$ for all i. We note that in this case a necessary and sufficient condition for A to be singular is that there exist nonsingular diagonal matrices D_1, D_2 such that $D_1 A D_2$ is a Z-matrix with nonnegative diagonal entries. Then we suggest an algorithm to determine whether this latter property is satisfied. This algorithm can be used in place of the one suggested recently by Farid [14, Theorem 4.1] in determining invertibility of a diagonally dominant matrix. We give some comments to Farid's work and also pose an open problem.

Our treatment also yields, as a by-product, information about eigenvalues (and the associated eigenvectors) that lie on the boundary of various spectrum inclusion regions for an irreducible matrix. For instance, an eigenvector associated with such an eigenvalue must have nonzero components; hence, such an eigenvalue must be of geometric multiplicity one. Our results explain and clarify the recent work or Zhang and Gu [25].

In the last part of the paper we give refinements of following theorem of Solov'ev [20, Theorem 1] for the irreducible matrix case.

THEOREM B. *Let A be an $n \times n$ matrix, and let r be an integer with $1 \leqslant r \leqslant n$. Assume that A satisfies both of the following conditions:*

(i) *For each $j = 1, 2, \dots, n$,*

$$|a_{jj}| > C_j^{(r-1)},$$

where $C_j^{r-1)}$ is the sum of the moduli of the $r-1$ largest off-diagonal entries in the jth column.

(ii) *For each set of r rows of A, the sum of the moduli of the diagonal entries in those rows is strictly greater than the sum of the moduli of all the off-diagonal entries in those rows.*

Then A is invertible.

2. AN AUXILIARY RESULT

For a positive integer n, we denote by $\langle n \rangle$ the set $\{1, 2, \dots, n\}$.

As we have mentioned in the introductory section, in our treatment we are going to encounter situations when a singular irreducible matrix A has an eigenvector x, with nonzero components, corresponding to the eigenvalue 0 such that $\mathfrak{M}(A)|x| = 0$, where $\mathfrak{M}(A)$ is the comparison matrix of A and $|x|$ is the modulus of x. In the following result we examine this latter property carefully.

Recall that a real matrix is called a *Z-matrix* if its off-diagonal entries are all nonpositive.

THEOREM 2.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix, $n \geqslant 2$. Consider the following conditions*:

(a) (i) *There exists a nonzero vector x all of whose components have equal moduli such that $Ax = 0$ and $\mathfrak{M}(A)|x| = 0$.*

(ii) *There exist unitary diagonal matrices D_1, D_2 such that D_1AD_2 is a Z-matrix with zero row sums.*

(b) (i) *There exists a nonzero vector x with nonzero components such that $Ax = 0$ and $\mathfrak{M}(A)|x| = 0$.*

(ii) *There exist nonsingular diagonal matrices D_1, D_2 such that D_1AD_2 is a Z-matrix with zero row sums.*

(c) *There exist nonsingular (or unitary) diagonal matrices D_1, D_2 such that D_1AD_2 is a Z-matrix with nonnegative diagonal entries.*

(d) (i) *Let $B = (b_{ij})$ be the $n \times n$ matrix given as follows: b_{ij} equals $-a_{ij}$ if $i = j$ and equals a_{ij} if $i \neq j$. Denote by b^i the ith row vector of B. Then for all pairs $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, we have*

$$|\langle b^i, b^j \rangle| = \langle |b^i|, |b^j| \rangle,$$

where we use $\langle x, y \rangle$ to denote the usual inner product of $\mathbb{C}^n$ between the vectors x and y.

(ii) *For all pairs $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, we have*

$$\left| \sum_{l \neq i, j} a_{il}\bar{a}_{jl} - a_{ij}\bar{a}_{jj} - a_{ii}\bar{a}_{ji} \right| = \sum_{l=1}^{n} |a_{il}|\,|a_{jl}|.$$

(e) *0 is an eigenvalue of A, and*

$$|a_{ii}| = R_i \qquad \textit{for each} \quad i \in \langle n \rangle.$$

Then conditions (i) *and* (ii) *given in* (a) (*also those in* (b) *and in* (d)) *are equivalent. Furthermore, the following logical implications hold:* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *and* (a) $\Rightarrow$ (e). *When A is irreducible and satisfies $|a_{ii}| = R_i$ for all $i \in \langle n \rangle$,* (a), (b), (c), *and* (e) *are all equivalent.*

Proof. We first establish the equivalence of conditions (i) and (ii) as given in (a), as well as those given in (b) and in (d).

(a) (i) $\Rightarrow$ (ii): Let $x = (x_1, \ldots, x_n)^T$ be a nonzero vector all of whose components have equal moduli such that $Ax = 0$ and $\mathfrak{M}(A)|x| = 0$. Then, for each $i \in \langle n \rangle$, we have

$$-a_{ii}x_i = \sum_{j \neq i} a_{ij}x_j \tag{2.2}$$

and

$$\sum_{j \neq i} |a_{ij}|\,|x_j| = |a_{ii}|\,|x_i| = \left|\sum_{j \neq i} a_{ij}x_j\right|; \tag{2.3}$$

hence the nonzero numbers among

$$a_{i1}x_1, \ldots, a_{i,i-1}x_{i-1}, a_{i,i+1}x_{i+1}, \ldots, a_{in}x_n$$

have the same argument as that of $-a_{ii}x_i$ (unless $a_{ii}x_i = 0$, in which case the numbers $a_{ij}x_j$, $j \in \langle n \rangle$, are all zero). Thus for each $i \in \langle n \rangle$, there exists a complex number w_i of modulus 1 such that the numbers $a_{ij}x_jw_i$, $j \in \langle n \rangle \setminus \{i\}$, are all nonpositive, the number $a_{ii}x_iw_i$ is nonnegative, and the sum of these n numbers is zero. Take D_1 and D_2 to be the unitary diagonal matrices given by $D_1 = \operatorname{diag}(w_1, \ldots, w_n)$ and $D_2 = \operatorname{diag}(x_1, \ldots, x_n)$. Then D_1AD_2 is a Z-matrix with zero row sums.

(a) (ii) $\Rightarrow$ (i): Suppose that there exist unitary diagonal matrices D_1, D_2 such that D_1AD_2 is a Z-matrix with zero row sums; say, $D_1 = \operatorname{diag}(w_1, w_2, \ldots, w_n)$ and $D_2 = \operatorname{diag}(x_1, x_2, \ldots, x_n)$. Then for each $i \in \langle n \rangle$, the numbers $a_{ij}x_jw_i$, $j \in \langle n \rangle \setminus \{i\}$, are nonpositive, $a_{ii}x_iw_i$ is nonnegative, and the sum of these n numbers is zero. Hence, Equations (2.2) and (2.3) are both satisfied for all $i \in \langle n \rangle$. Thus the vector $x = (x_1, \ldots, x_n)^T$ fulfills the requirement of (i).

Note that in the above proof of (a) (i) $\Rightarrow$ (ii), the condition that the components of x have equal moduli is used only once in order to insure that the matrix D_2 will be unitary. Thus the same argument also establishes (b) (i) $\Rightarrow$ (ii). Similarly, the argument for (a) (ii) $\Rightarrow$ (i) also works for (b) (ii) $\Rightarrow$ (i).

Conditions (i) and (ii) in (d) are also equivalent, as can be seen by rewriting condition (i) in terms of the entries of A.

The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (d)(i): Suppose that there exist nonsingular diagonal matrices U, V such that $A = UXV$, where $X = (x_{ij})$ is a Z-matrix with nonnegative diagonal entries. Let $U = \operatorname{diag}(u_1, \dots, u_n)$ and $V = \operatorname{diag}(v_1, \dots, v_n)$. Consider any pair $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$. Using the definition of B as given in (d)(i), we have

$$\begin{aligned}\langle b^i, b^j \rangle &= \sum_{k \neq i, j} (u_i x_{ik} v_k) \overline{(u_j x_{jk} v_k)} + [u_i(-x_{ii}) v_i] \overline{[u_j x_{ji} v_i]} \\ &\quad + [u_i x_{ij} v_j] \overline{[u_j(-x_{jj}) v_j]} \\ &= u_i \bar{u}_j \left\{ \sum_{k \neq i, j} x_{ik} x_{jk} |v_k|^2 - x_{ii} x_{ji} |v_i|^2 - x_{ij} x_{jj} |v_j|^2 \right\}.\end{aligned}$$

By our assumption on X, it is clear that each term inside the braces is nonnegative. Now we can readily write out the expression for $\langle |b^i|, |b^j| \rangle$ and obtain

$$|\langle b^i, b^j \rangle| = \langle |b^i|, |b^j| \rangle.$$

(a)(ii) $\Rightarrow$ (e): When condition (a)(ii) is satisfied, it is clear that the matrix $D_1 A D_2$ has the property that the modulus of each of its diagonal entries equals the sum of moduli of all off-diagonal entries in the same row. Since the diagonal matrix D_2 is unitary, A also possesses the same property. Since 0 is an eigenvalue of $D_1 A D_2$, clearly it is also an eigenvalue of A.

Last part: It is clear that in condition (c) it makes no difference whether we take the diagonal matrices D_1, D_2 to be nonsingular or to be unitary. When $|a_{ii}| = R_i$ holds for all $i \in \langle n \rangle$, using the above argument for (a)(ii) $\Rightarrow$ (e), we readily establish the implication (c) $\Rightarrow$ (a)(ii).

It remains to show that when A is irreducible, condition (e) implies condition (a)(i). The assumption that $|a_{ii}| = R_i$ holds for all $i \in \langle n \rangle$ means that 0 lies on the boundary of each Geršgorin disk of A. Let $x = (x_1, \dots, x_n)^T$ be an eigenvector of A corresponding to 0. Since A is irreducible, it is well known that in this case we have $|x_1| = |x_2| = \cdots = |x_n|$ (see, for instance, [16, Theorem 6.2.8]), and also that $|a_{ii}|\,|x_i| = \sum_{j \neq i} |a_{ij}|\,|x_j|$ holds for all i (see [16, Lemma 6.2.3 and its proof]). Hence, x is the desired vector that satisfies the requirement of condition (a)(i). ■

REMARK 2.4. In the last part of Theorem 2.1 we have not stated the result in the strongest possible form. Actually, as can be seen from the proof, we have the implication (e) $\Rightarrow$ (a) when A is irreducible, and also the implication (c) $\Rightarrow$ (a) when $|a_{ii}| = R_i$ holds for all $i \in \langle n \rangle$. Without the irreducibility assumption on A, we do not even have the implication (e) $\Rightarrow$ (d). The following matrix A can serve as a counterexample:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

3. DIAGONALLY DOMINANT MATRICES

In this section we make a digression and take a look at diagonally dominant matrices.

Recall that an $n \times n$ matrix $A = (a_{ij})$ is said to be *diagonally dominant* if $|a_{ii}| \geqslant R_i$ for all $i \in \langle n \rangle$, *strictly diagonally dominant* if all the inequalities are strict, and *irreducibly diagonally dominant* if it is irreducible and diagonally dominant, and $|a_{ii}| > R_i$ holds for at least one i. According to the Levy-Desplanques theorem and Taussky's theorem, if A is strictly diagonally dominant or irreducibly diagonally dominant, then A is invertible.

Recently in [14, Theorem 4.1] Farid gave a criterion to determine whether a (reducible) diagonally dominant matrix (with nonzero diagonal entries) is singular. In essence, his criterion consists of two parts. The first part says that a necessary condition for a diagonally dominant matrix A with nonzero diagonal entries to be singular is that there exists an $n \times n$ permutation matrix P and some integer r with $2 \leqslant r \leqslant n$ such that

$$P^T A P = \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is an $r \times r$ irreducible diagonally dominant matrix with nonstrict diagonally dominant rows (i.e., the modulus of each of its diagonal entries equals the corresponding deleted absolute row sum). The second part gives a criterion to check whether the irreducible block A_{11} is singular, and is given in terms of the principal arguments of the nonzero entries of A_{11}, assuming that A_{11} is already in a certain specific form (see conditions 2.1 and 2.2 in Theorem 4.1 of [14]). In order to obtain the first part of the criterion, much work is done in Sections 2 and 3 of [14] on relationships among the rows of a diagonally dominant singular matrix. Here we note two points about the first part of the criterion. First, for the purpose of checking singularity it is not put in an effective form. Second, there is an easier way to derive it. We are going to explain.

Let A be a diagonally dominant matrix with nonzero diagonal entries. To determine whether A is singular or not, we can look at its Frobenius normal form and check whether at least one of the irreducible blocks (which are necessarily diagonally dominant) is singular. If $A[\alpha]$ is an irreducible block (with rows and columns indexed by $\alpha \subseteq \langle n \rangle$), we call it a *final block* if there do not exist $i \in \alpha$ and $j \in \langle n \rangle \setminus \alpha$ such that $a_{ij} \neq 0$. All irreducibly blocks which are not final are clearly irreducibly diagonally dominant and hence are invertible. Thus, we need consider only the final irreducible blocks of A, and indeed only those with nonstrict diagonally dominant rows. In order for A to be singular, it is necessary that A has a final irreducible block with nonstrict diagonally dominant rows. By a suitable permutation similarity we can always bring a into A Frobenius normal form with such a block (if it exists) as its first irreducible diagonal block. In general, A may have several final irreducible blocks with nonstrict diagonally dominant rows, and we do not know *a priori* which of them is singular and which is not. This explains the first part of Farid's criterion and also the fact that it is not put in an effective form.

The contribution of Farid's criterion lies in its second part—a criterion for an irreducible diagonally dominant matrix with nonstrict diagonally dominant rows to be singular. In order to apply this criterion to an $n \times n$ irreducible matrix $A = (a_{ij})$, we need to bring A (by a suitable permutation similarity) into a form with the following property: for every $i \in \langle n-1 \rangle$ there exists an integer $p_i \in \{i+1, \dots, n\}$ such that $a_{ip_i} \neq 0$. Corollary 4.1 (which, in turn, depends on Lemma 4.2) of [14] guarantees that this can always be done. Clearly, the above property of an irreducible matrix can be reformulated as a result about the relabeling of a strongly connected digraph. Below we offer a simple (algorithmic) proof for this graph-theoretic result.

LEMMA C. *Let Γ be a strongly connected digraph with $n \geqslant 2$ vertices. We can always label the vertices of Γ by integers taken from $\langle n \rangle$ in such a way that for any vertex $i \in \langle n-1 \rangle$, there is a vertex $p_i \in \{i+1, \dots, n\}$ such that (i, p_i) is an arc of Γ.*

Proof. To begin with, we take any nontrivial cycle γ of Γ, of length, say m, and label its vertices by integers $1, 2, \dots, m$, in the order of the cycle. With this labeling, the subdigraph of Γ induced by these m vertices clearly satisfies our desired property. If $m = n$, we are done.

Suppose that at some intermediate stage we have labeled the vertices of a strongly connected proper subdigraph Γ_1 of Γ in such a way that the desired property is satisfied by Γ_1. Let Γ have p vertices with $2 \leqslant p < n$. Then we can find a (directed simple) path of length $q+1$ $(q \geqslant 2)$ that begins and ends at Γ_1 and such that the intermediate vertices (q of them) all lie outside Γ_1. Now label the intermediate vertices of this path respectively by $1, 2, \dots, q$ in the order in which the path is traversed. Also relabel each of the vertices of Γ_1 by adding q to the previous label. The subdigraph Γ_2 of Γ induced by the vertices of Γ_1 and the q intermediate vertices of the above path is clearly strongly connected. Furthermore, with this labeling Γ_2 satisfies the desired property. If $p+q=n$, we are done. Otherwise, continue the process. After a finite number of steps, we can obtain a labeling of the vertices of Γ with the desired property. ∎

According to the last part of our Theorem 2.1, if A is an irreducible diagonally dominant matrix with nonstrict diagonally dominant rows, then the singularity of A is equivalent to condition (c) of Theorem 2.1. As can be readily seen [cf. the proof of Theorem 2.1(a)(i) $\Rightarrow$ (ii)], when A has nonzero diagonal entries, the latter condition is, in turn, equivalent to the following property: There exist complex numbers $x_1, x_2, \dots, x_n$ all of moduli one (or nonzero) such that for each $i \in \langle n \rangle$, the nonzero numbers among

$$a_{i1}x_1, a_{i2}x_2, \dots, a_{i,i-1}x_{i-1}, a_{i,i+1}x_{i+1}, \dots, a_{in}x_n$$

have the same argument as that of $-a_{ii}x_i$.

We are going to derive an algorithm to determine whether the preceding property is fulfilled by an irreducible matrix A with nonzero diagonal entries. Our algorithm, which does not rely on Corollary 4.1 of [14], can be used in place of Farid's criterion in determining whether an irreducible diagonally dominant matrix is singular.

Let A be an $n \times n$ irreducible matrix, $n \geqslant 2$, with nonzero diagonal entries. For convenience, we work with the $n \times n$ matrix B given as follows: b_{ij} equals $-a_{ij}$ if $i=j$ and equals a_{ij} if $i \neq j$. Clearly B has the same digraph as A. For each $i \in \langle n \rangle$, we denote by $\Gamma^+(P_i)$ the set of all vertices P_j, $j \neq i$, for which there is an arc in $\Gamma(B)$ from P_i to P_j. Let t be the length of the longest path in $\Gamma(B)$ that begins with the vertex P_1. For each $k \in \langle t \rangle$, we use $\mathfrak{D}_k$ to denote the set of all indices $j \in \langle n \rangle \setminus \{1\}$ for which the length of the shortest path in $\Gamma(B)$ from P_1 and P_j equals k. We also set $\mathfrak{D}_0 = \{1\}$.

It is clear that if the desired n-tuple of complex numbers $(x_1, \dots, x_n)$ exists, then it must be unique up to a scalar multiple (of modulus one). In this case we can always choose it in such a way that the nonzero numbers among

$$b_{11}x_1, b_{12}x_2, \dots, b_{1n}x_n$$

are all positive. So we proceed as follows.

(1) Choose the (unique) complex number x_1 with modulus 1 such that $b_{11}x_1 > 0$.

(2) For each $i \in \mathfrak{D}_1$, choose the complex number x_i of modulus one such that $b_{1i}x_i > 0$. (In order for the nonzero numbers among $b_{11}x_1, b_{12}x_2, \dots, b_{1n}x_n$ all to be positive, we have to choose x_i for all $i \in \mathfrak{D}_1$ in this way.)

(k) At a general step, for $2 \leqslant k \leqslant t$, suppose that we have already determined the complex numbers x_j with moduli one for all $j \in \bigcup_{l=0}^{k-1}\mathfrak{D}_l$. For each $j \in \mathfrak{D}_k$, in order to determine x_j, choose some $i \in \mathfrak{D}_{k-1}$ such that $b_{ij} \neq 0$. (There may be more than one such i, but just choose one.) Take x_j to be the complex number of modulus one such that $b_{ii}x_i$ and $b_{ij}x_j$ have the same argument.

In this way, after a finite number of steps, we arrive at an n-tuple $(x_1, x_2, \dots, x_n)$ of complex numbers, all of moduli one. Note that if the desired n-tuple exists and is chosen to satisfy steps 1 and 2 (so that the nonzero numbers among $b_{11}x_1, b_{12}x_2, \dots, b_{1n}x_n$ are all positive), then necessarily it is equal to $(x_1, x_2, \dots, x_n)$. As a final step, we check whether it is indeed true that, for each $i \in \langle n \rangle$, the nonzero numbers among

$$b_{11}x_1, b_{12}x_2, \dots, b_{1n}x_n$$

have the same argument.

If Γ is a digraph with vertex set $\{P_1, \dots, P_n\}$, then by the *undirected graph of* Γ we mean the undirected graph G with the same vertex set such that there is an edge in G between P_i and P_j if and only if $i \neq j$ and (P_i, P_j) and (P_j, P_i) are both arcs of Γ. A matrix $A = (a_{ij})$ is said to be *combinatorially symmetric* if $a_{ij} \neq 0$ whenever $a_{ji} \neq 0$.

If A is a combinatorially symmetric irreducible matrix with nonzero diagonal entries such that the undirected graph of $\Gamma(A)$ is a tree, then a moment's thought upon our above algorithm will show that it is always possible to choose an n-tuple $(x_1, x_2, \dots, x_n)$ of nonzero complex numbers such that for each $i \in \langle n \rangle$, the nonzero numbers among

$$a_{i1}x_1, a_{i2}x_2, \dots, a_{i,i-1}x_{i-1}, a_{i,i+1}x_{i+1}, \dots, a_{in}x_n$$

have the same argument as that of $-a_{ii}x_i$. Thus, for such a matrix A, if it is diagonally dominant with nonstrict diagonally dominant rows, then it must be singular.

Either our above algorithm or Farid's criterion is not an entirely satisfactory result for invertibility in that both criteria are not given explicitly in terms of the entries of the matrix under consideration. After all, to decide whether a matrix is singular, we can always reduce it to a row echelon form by elementary row operations and count its rank.

When A is an irreducible diagonally dominant matrix with nonstrict diagonally dominant rows, condition (d)(ii) of Theorem 2.1 provides an explicit, readily checkable condition necessary for A to be singular. One may ask whether in this case it is also a sufficient condition. Unfortunately, the answer turns out to be no, as can be illustrated by the following matrix:

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & 2i & 0 & 2i \\ 0 & 0 & 1+i & \sqrt{2} \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Nevertheless, we have the following positive result in this direction.

THEOREM 3.1. *Let A be an $n \times n$ matrix, $n \geqslant 2$, that satisfies both of the following conditions*:

(a) *For each $i \in \langle n \rangle$, we have $|a_{ii}| = R_i$.*

(b) *There exists some $k \in \langle n \rangle$ such that the entries in the kth row of A are all nonzero. Furthermore, for each $j \in \langle n \rangle \setminus \{k\}$, we have*

$$\left| \sum_{l \neq k,j} a_{kl}\bar{a}_{jl} - a_{kj}\bar{a}_{jj} - a_{kk}\bar{a}_{jk} \right| = \sum_{l=1}^{n} |a_{kl}|\,|a_{jl}|.$$

Then 0 is an eigenvalue of A.

Proof. Let $B = (b_{ij})$ be the $n \times n$ matrix given as follows: b_{ij} equals a_{ij} for $i \neq j$, and equals $-a_{ij}$ for $i = j$. Denote by b^i the ith row vector of B. Then the equalities in condition (b) amount to saying that

$$|\langle b^k, b^j \rangle| = \langle |b^k|, |b^j| \rangle$$

for all $j \in \langle n \rangle \setminus \{k\}$. Since the entries in the kth row of B are all nonzero, we can find complex numbers $x_1, \dots, x_n$ all of moduli one such that

$(b_{k1}x_1, b_{k2}x_2, \dots, b_{kn}x_n)^T$ is a positive vector. For each $i \in \langle n \rangle$, denote by $\tilde{b}_i$ the vector $(b_{i1}x_1, \dots, b_{in}x_n)^T$. Then, since the x_i's are all of moduli one, we have

$$\langle b^k, b^j \rangle = \langle \tilde{b}^k, \tilde{b}^j \rangle \quad \text{and} \quad \langle |b^k|, |b^j| \rangle = \langle |\tilde{b}^k|, |\tilde{b}^j| \rangle$$

and hence

$$|\langle \tilde{b}^k, \tilde{b}^j \rangle| = \langle |\tilde{b}^k|, |\tilde{b}^j| \rangle$$

for all $j \in \langle n \rangle \setminus \{k\}$. But $\tilde{b}^k$ is a positive vector, so it follows that for each $i \in \langle n \rangle$, the nonzero numbers among

$$b_{i1}x_1, b_{i2}x_2, \dots, b_{in}x_n$$

have the same argument. In other words, condition (c) of Theorem 2.1 is fulfilled by A. In view of condition (a), by Remark 2.4 A satisfies condition (a) of Theorem 2.1, and hence 0 is eigenvalue of A. ■

EXAMPLE 3.2. Let $A = (a_{ij})$ be a 5×5 irreducible matrix with the following zero-nonzero pattern:

$$\begin{bmatrix} * & * & * & 0 & 0 \\ 0 & * & 0 & * & * \\ * & 0 & * & * & * \\ 0 & * & 0 & * & 0 \\ * & * & 0 & 0 & * \end{bmatrix},$$

where we use $*$ to indicate a nonzero position and a 0 for a zero position. Suppose that A satisfies $|a_{ii}| = R_i$ for all i, and also condition (d) of Theorem 2.1. We are going to show that A must be a singular matrix.

Let $B = (b_{ij})$ denote the 5×5 matrix given as follows: b_{ij} equals a_{ij} for $i \neq j$, and equals $-a_{ij}$ for $i = j$. Also let b^i denote the ith row vector of B. First, choose complex numbers x_1, x_2, and x_3 of moduli one such that the numbers $b_{11}x_1, b_{12}x_2, b_{13}x_3$ are all positive. Then choose complex numbers x_4 and x_5, again of moduli one, such that $b_{22}x_2, b_{24}x_4$, and $b_{25}x_5$ have the same argument. For $i = 1, \dots, 5$, let $\tilde{b}^i = (b_{i1}x_1, \dots, b_{i5}x_5)^T$. We contend that with the above choice of $x_1, \dots, x_5$, the nonzero components of each vector $\tilde{b}^i$ have the same argument. Once this is done, as noted before, condition (c) of Theorem 2.1 is satisfied by A, and hence A is singular.

Since A satisfies condition (d) of Theorem 2.1, we have

$$|\langle \tilde{b}^1, \tilde{b}^5\rangle| = |\langle b^1, b^5\rangle| = \langle |b^1|, |b^5|\rangle = \langle |\tilde{b}^1|, |\tilde{b}^5|\rangle.$$

But by our choice of x_1, x_2, the complex numbers $b_{11}x_1$ and $b_{12}x_2$ are real and positive; hence $b_{51}x_1$ and $b_{52}x_2$ have the same argument. Similarly, from the condition $|\langle b^2, b^5\rangle| = \langle |b^2|, |b^5|\rangle$ and the fact that $b_{22}x_2$ and $b_{25}x_5$ have the same argument, we infer that $b_{52}x_2$ and $b_{55}x_5$ have the same argument. Hence, the nonzero components of the vector $\tilde{b}^5$ have the same argument. In a similar way, using the condition $|\langle b^2, b^4\rangle| = \langle |b^2|, |b^4|\rangle$, we infer that the nonzero components of the vector $\tilde{b}^4$ have the same argument. Finally, using the conditions $|\langle b^1, b^3\rangle| = \langle |b^1|, |b^3|\rangle, |\langle b^2, b^3\rangle| = \langle |b^2|, |b^3|\rangle$ and $|\langle b^3, b^5\rangle| = \langle |b^3|, |b^5|\rangle$, we also conclude that the nonzero components of the vector $\tilde{b}^3$ have the same argument. This establishes our contention and hence our claim on the matrix A.

PROBLEM 3.3. Identify all $n \times n$ irreducible zero-nonzero patterns C with the property that for any matrix $A = (a_{ij})$ with the same zero-nonzero pattern as C, if A is diagonally dominant with nonstrict diagonally dominant rows and satisfies condition (d) of Theorem 2.1, then A is a singular matrix.

4. VARIANTS OF BRUALDI'S THEOREMS

The following problem was much studied in the 1960s and 1970s: Given an $n \times n$ matrix $A = (a_{ij})$, determine those n-tuples $\tau = (\tau_1, \ldots, \tau_n)$ of nonnegative real numbers [depending on the moduli of the $n(n-1)$ off-diagonal entries of A] such that every eigenvalue of A lies in at least one of the n circular disks

$$\{z : |z - a_{ii}| \leqslant \tau_i\}, \qquad i = 1, 2, \ldots, n.$$

In Theorem 4.1 below we are going to collect a list of known or partly known equivalent conditions for such n-tuples τ. Using the theory of M-matrices and Brualdi's theorem [6, Theorem 2.9], we offer a self-contained proof. Before we come to the result, we recall some definitions and introduce a new notation.

An $n \times n$ real matrix A is called an *M-matrix* if there exists an $n \times n$ nonnegative matrix P and some nonnegative real number λ such that

$A = \lambda I - P$ and $\lambda \geqslant \rho(P)$ (the spectral radius of P); if $\lambda > \rho(P)$, we call A a *nonsingular M-matrix.*

For an $n \times n$ matrix $A = (a_{ij})$ and an n-tuple $\tau(\tau_1, \dots, \tau_n)$ of nonnegative real numbers, we denote by $\mathfrak{M}^{\tau}(A) = (m_{ij})$ the matrix given by

$$m_{ij} = \begin{cases} \tau_i & \text{for } i = j, \\ -|a_{ij}| & \text{for } i \neq j. \end{cases}$$

Clearly $\mathfrak{M}^{\tau}(A)$ is always a Z-matrix and its definition is independent of the diagonal entries of A. In particular, when $\tau = (|a_{11}|, |a_{22}|, \dots, |a_{nn}|)$, $\mathfrak{M}^{\tau}(A)$ becomes $\mathfrak{M}(A)$, the comparison matrix of A.

THEOREM 4.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix. Consider the following conditions on an n-tuple $\tau = (\tau_1, \dots, \tau_n)$ of nonnegative real numbers*:

(a) *$\mathfrak{M}^{\tau}(A)$ is a (possibly singular) M-matrix.*

(b) *For any $n \times n$ complex matrix $B = (b_{ij})$ satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, if*

$$|b_{ii}| > \tau_i \qquad \textit{for each} \quad i \in \langle n \rangle,$$

then B is invertible.

(c) *For each $n \times n$ complex matrix $B = (b_{ij})$ satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in (p[n\rangle$, $i \neq j$, each eigenvalue of B lies in the union of the n disks*

$$\Delta_k = \{z \in \mathbb{C} : |z - b_{kk}| \leqslant \tau_k\}, \qquad k = 1, 2, \dots, n.$$

(d) *There exists a vector $x = (x_1, \dots, x_n)^T$ with positive components such that for each $i \in \langle n \rangle$*

$$\tau_i \geqslant \frac{1}{x_i} \sum_{j \neq 1} |a_{ij}| x_j.$$

(e) *For each $n \times n$ complex matrix $B = (b_{ij})$ satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, if*

$$\prod_{P_i \in \gamma} |b_{ii}| \geqslant \prod_{P_i \in \gamma} \tau_i$$

for all nontrivial cycles γ of $\Gamma(A)$, with strict inequality for at least one such cycle, then B is invertible.

(f) *For each $n \times n$ complex matrix $B = (b_{ij})$ satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, the eigenvalues of B lie in the region $\cup_\gamma D_\gamma$, where $D_\gamma = \{z : \prod_{P_i \in \gamma} |z - b_{ii}| \leqslant \prod_{P_i \in \gamma} \tau_i\}$ and the union is taken over all nontrivial cycles γ of $\Gamma(B)$.*

Then conditions (a), (b), and (c) are equivalent. Each of the conditions (d), (e), *and* (f) *is sufficient for* (a)–(c), *and is equivalent to them when A is irreducible.*

Proof. (a) $\Rightarrow$ (b): If $B = (b_{ij})$ is an $n \times n$ matrix satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, and $\langle b_{ii}| > \tau_i$ for each i, then $\mathfrak{M}(B)$ is equal to $\mathfrak{M}^\tau(A) + D$ for some diagonal matrix D with positive entries, and hence is a nonsingular M-matrix, as $\mathfrak{M}^\tau(A)$ is an M-matrix (see [2, Theorem 6.4.6, condition (A_3)]). Thus $\mathfrak{M}(B)$, and hence B, being diagonally similar to a strictly diagonally dominant matrix (see [2, Theorem 6.2.3, condition (M_{36})]), is invertible.

(b) $\Rightarrow$ (a): Condition (b) clearly implies that $\mathfrak{M}^\tau(A) + D$ is invertible for each diagonal matrix D with positive diagonal entries; hence, $\mathfrak{M}^\tau(A)$ is an M-matrix.

The equivalence of conditions (b) and (c) is obvious.

Note that for any positive vector $x = (x_1, \ldots, x_n)^T$, we have $\mathfrak{M}^\tau(A)x \geqslant 0$ if and only if $\tau_i \geq (1/x_i)\sum_{j \neq i} |a_{ij}|x_j$ for each $i \in \langle n \rangle$.

(d) $\Rightarrow$ (a): When condition (d) is satisfied, there exists a positive vector x such that $\mathfrak{M}^\tau(A)x \geqslant 0$. But $\mathfrak{M}^\tau(A)$ is a Z-matrix; hence it is an M-matrix (see [2, Exercise 6.4.14]).

The implication (e) $\Rightarrow$ (f) is clear, because condition (f) is equivalent to the following:

(f′) For any $n \times n$ matrix B satisfying $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, if

$$\prod_{P_i \in \gamma} |b_{ii}| > \prod_{P_i \in \gamma} \tau_i$$

for all nontrivial cycles γ of $\Gamma(B)$, then B is invertible.

(f′) $\Rightarrow$ (a): By condition (f′), for any diagonal matrix D with positive diagonal entries, the matrix $\mathfrak{M}^\tau(A) + D$ is invertible; hence $\mathfrak{M}^\tau(A)$ is an M-matrix.

Now we suppose, in addition, that A is irreducible.

(a) $\Rightarrow$ (d): Since $\mathfrak{M}^\tau(A)$ is an irreducible M-matrix, there exists a positive vector x such that $Ax \geqslant 0$ (see [2, Theorem 6.2.3, condition I_{27}) and Theorem 6.4.16]), hence condition (d) is satisfied.

(d) $\Rightarrow$ (e): Let $x = (x_1, \dots, x_n)^T$ be a positive vector that satisfies condition (d). Let $A^{(1)} = D^{-1}AD$, where D is the diagonal matrix $\operatorname{diag}(x_1, \dots, x_n)$. Denote by $R_i^{(1)}$ the ith deleted absolute row sum of $A^{(1)}$. Then condition (d) amounts to saying that $\tau_i \geqslant R_i^{(1)}$ for each $i \in \langle n \rangle$. Let B be an arbitrary $n \times n$ matrix that satisfies $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$, $i \neq j$, and

$$\prod_{P_i \in \gamma} |b_{ii}| \geqslant \prod_{P_i \in \gamma} \tau_i$$

for all nontrivial cycles γ of $\Gamma(B)$, with strict inequality for at least one such cycle. Then we have

$$\prod_{P_i \in \gamma} |b_{ii}| \geqslant \prod_{P_i \in \gamma} R_i^{(1)}$$

for all nontrivial cycles γ of $\Gamma(B)$, with strict inequality for at least one such cycle. Note that b_{ii} equals the ith diagonal entry of $D^{-1}BD$, and also that $R_i^{(1)}$ equals the ith deleted absolute row sum of $D^{-1}BD$. An application of Brualdi's theorem [6, Theorem 2.9] to the irreducible matrix $D^{-1}BD$ yields that $D^{-1}BD$, and hence B, is invertible. ■

A few remarks about the conditions of Theorem 4.1 are in order. When A is irreducible, the implication (c) $\Rightarrow$ (d) was due to Ky Fan [13, Theorem], who gave an elegant, but tricky proof that depends on the use of Perron-Frobenius theorem for an irreducible nonnegative matrix. The equivalence of conditions (a), (b), and (c), and also (d) in the irreducible case, was mentioned (without proof) in [10, Proposition 1]. The implication (c) $\Rightarrow$ (f) (for an irreducible matrix A) was due to Brualdi [6, Theorem 2.12]. Our proof of (d) $\Rightarrow$ (e) is similar to his proof of this implication; both rely on the fact that condition (d) is equivalent to the condition that for each $i \in \langle n \rangle$, $\tau_i \geq R_i^{(1)}$, where $R_i^{(1)}$ is the ith deleted absolute row sum of the matrix $D^{-1}AD$, where $D = \operatorname{diag}(x_1, \dots, x_n)$. The implication (f) $\Rightarrow$ (d) (for an irreducible matrix A) is also the content of [6, Theorem 2.14].

If $\tau = (R_1^{\alpha}C_1^{1-\alpha}, R_2^{\alpha}C_2^{1-\alpha}, \dots, R_n^{\alpha}C_n^{1-\alpha})$, where $0 \leqslant \alpha \leqslant 1$, then by Ostrowski's theorem τ is an n-tuple that satisfies the equivalent conditions of Theorem 4.1. Many other examples of such n-tuples τ can also be found in the literature (see [1, 10, 11, 13, 15]). Indeed, they are sometimes given in terms of the concept of G-functions (see [10, 11, 15]), but for our purposes there is no need to introduce such a function concept.

REMARK 4.2. Since a Z-matrix is an M-matrix if and only if each of its maximal irreducible principal submatrices is an M-matrix, it is not difficult to show that when A is weakly irreducible, conditions (f) and (a) of Theorem 4.1 are equivalent. Then the following condition (d′) [in place of condition (d)] is also another equivalent condition:

(d′) There exists a positive vector $x = (x_1, \dots, x_n)^T$ such that for each $i \in \langle n \rangle$,

$$\tau_i \geqslant \frac{1}{x_i} \sum_{j \in S_i \setminus \{i\}} |a_{ij}| x_j,$$

where $S_i = \{j \in \langle n \rangle : P_i$ and P_j belong to the same strongly connected component of $\Gamma(A)\}$.

In passing, we also note the following result (see [23]):

REMARK 4.3. For any $n \times n$ irreducible matrix A, the condition that $\mathfrak{M}(A)$ is a singular M-matrix is equivalent to the existence of a positive vector $x = (x_1, \dots, x_n)^T$ such that 0 lies on the intersection of the n Geršgorin circles of $D^{-1}SD$, where D is the diagonal matrix $\operatorname{diag}(x_1, x_2, \dots, x_n)$. Furthermore, when the equivalent conditions are satisfied, 0 is an eigenvalue of some $n \times n$ matrix $B = (b_{ij})$ with the property that $b_{ii} = a_{ii}$ and $|b_{ij}| = |a_{ij}|$ for all $i, j \in \langle n \rangle$; hence 0 belongs to the minimal Geršgorin set of A (i.e. the intersection of the Geršgorin regions of all matrices that are diagonally similar to A).

In Theorem 4.5 we are going to answer the question of how to relax the sufficient conditions given in Theorem A for invertibility when the matrix under consideration is irreducible. We give our answers in a more general context with the n-tuple $(R_1^\alpha C_1^{1-\alpha}, \dots, R_n^\alpha C_n^{1-\alpha})$ replaced by an n-tuple $\tau = (\tau_1, \dots, \tau_n)$ of positive numbers that satisfies the equivalent conditions of Theorem 4.1.

An undirected graph G with vertex set $\{P_1, \dots, P_n\}$ is called a *star* if there exists a vertex P_r, referred to as the *center* of the star, such that there are edges between P_r and every other vertex of G, but G has no other edges.

We need Lemma 4.4 in the proof of Theorem 4.5.

LEMMA 4.4. *Let Γ be a strongly connected digraph with vertex set $\{P_1, P_2, \dots, P_n\}$, $n \geqslant 2$. To each vertex P_i let there be given a positive real number w_i, and suppose that we have $w_i w_j \leqslant 1$ for all pairs $(i, j) \in \langle n \rangle \times$*

$\langle n \rangle$, $i \neq j$, with strict inequality for at least one such pair. Then exactly one of the following holds:

(a) *There exists a pair $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, for which the vertices P_i, P_j lie on a common cycle of Γ, such that $w_i w_j < 1$.*

(b) *$n \geqslant 3$, and the undirected graph of Γ is a star. Furthermore, if P_r is the center of the star, then we have*

$$1 < w_r = 1/w_j \qquad \text{for each} \quad j \in \langle n \rangle \setminus \{r\}.$$

Proof. Suppose that condition (a) is not satisfied. Then clearly we have $n \geqslant 3$ and $w_i w_j = 1$ whenever P_i, P_j $(1 \neq j)$ lie on a common cycle of Γ. Assume that there exists some $k \in \langle n \rangle$ such that $w_k = 1$. By the strong connectedness of Γ, clearly there exists a vertex P_l different from P_k such that P_l and P_k lie on a common cycle. For this pair (i, k) we have $w_{ik} = 1$; hence we also have $w_i = 1$. Since Γ is strongly connected, by continuing this argument, we would obtain $w_j = 1$ for all $j \in \langle n \rangle$. This contradicts the hypothesis that we have strict inequality $w_i w_j < 1$ for at least one pair $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$. This shows that $w_i \neq 1$ for all $i \in \langle n \rangle$.

Because $w_i w_j = 1$ whenever P_i, P_j $(i \neq j)$ lie on a common cycle, clearly there exists at least one $r \in \langle n \rangle$ such that $w_r > 1$, and in view of our hypothesis there can only be one such r. Consequently, there cannot exist an arc between a pair of distinct vertices P_i, P_j, both different from P_r. Hence, by the strong connectedness of Γ, for any $j \in \langle n \rangle \setminus \{r\}$, there exist arcs from P_r to P_j and from P_j to P_r, and thus $w_j = 1/w_r$. This shows that the undirected graph of Γ is a star with center at P_r and with the desired properties. ∎

THEOREM 4.5. *Let $A = (a_{ij})$ be an $n \times n$ irreducible matrix, $n \geqslant 2$. Let $\tau = (\tau_1, \dots, \tau_n)$ be an n-tuple of positive numbers such that $\mathfrak{M}^{\tau}(A)$ is an M-matrix (and hence satisfies the equivalent conditions of Theorem 4.1). Consider the following conditions:*

(a) *$|a_{ii}| \geq \tau_i$ for all $i \in \langle n \rangle$, with strict inequality for at least one i.*

(b) *$|a_{ii}|\,|a_{jj}| \geqslant \tau_i \tau_j$ for all pairs $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, with strict inequality for at least one such pair. In addition, there does not exist an index $r \in \langle n \rangle$ such that the off-diagonal entries in the rth row and rth column of A are all nonzero and all other off-diagonal entries of A are zero.*

(c) *$|a_{ii}|\,|a_{jj}| \geqslant \tau_i \tau_j$ for all pairs $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, for which P_i, P_j, lie on a common cycle of $\Gamma(A)$, with strict inequality for at least one such pair.*

(d) *$\prod_{P_i \in \gamma} |a_{ii}| \geqslant \prod_{P_i \in \gamma} \tau_i$ for all nontrivial cycles γ of $\Gamma(A)$, with strict* *inequality for at least one such cycle.*

(e) $\prod_{P_i \in \gamma} |a_{ii}| = \prod_{P_i \in \gamma} \tau_i$ *for all nontrivial cycles* γ *of* $\Gamma(A)$. *Furthermore, there exists a pair* $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, *such that*

$$\sum_{l=1}^{n} |a_{il}|\,|a_{jl}| > \left| \sum_{l \neq i, j} a_{il}\bar{a}_{jl} - a_{ij}\bar{a}_{jj} - a_{ii}\bar{a}_{ji} \right|.$$

Then the following implications hold: (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c), *and* (c) $\Rightarrow$ (d). *Furthermore, conditions* (d) *and* (e) *are each sufficient for invertibility of* A.

Proof. (a) $\Rightarrow$ (c): Obvious.

(b) $\Rightarrow$ (c): Apply Lemma 4.4 with $\Gamma = \Gamma(A)$ and $w_i = \tau_i/|a_{ii}|$ for $i = 1, 2, \ldots, n$.

(c) $\Rightarrow$ (d): Not difficult (cf. the proof of [25, Theorem 1]).

By Theorem 4.1 (a) $\Rightarrow$ (e) (in the irreducible case) it is clear that condition (d) is sufficient for invertibility of A. It remains to show that condition (e) is also sufficient for invertibility of A.

Since $\mathfrak{M}^{\tau}(A)$ is an M-matrix, condition (d) of Theorem 4.1 is satisfied. Hence, there exists a diagonal matrix $D = \mathrm{diag}(x_1, \ldots, x_n)$ with positive diagonal entries such that

$$\tau_i \geqslant R_i^{(1)} \qquad \text{for each} \quad i \in \langle n \rangle,$$

where $R_i^{(1)}$ is the ith deleted absolute row sum of $A^{(1)}$, and $A^{(1)} = D^{-1}AD$.

Suppose that condition (e) holds, but A is singular. Denote the (i, i) entry of $A^{(1)}$ by $a_{ii}^{(1)}$. Then for all nontrivial cycles γ of $\Gamma(A^{(1)})$, we have

$$\prod_{P_i \in \gamma} |a_{ii}^{(1)}| = \prod_{P_i \in \gamma} |a_{ii}| = \prod_{P_i \in \gamma} \tau_i \geqslant \prod_{P_i \in \gamma} R_i^{(1)}. \tag{4.6}$$

Since A is singular, so is $A^{(1)}$. Hence by [6, Theorem 2.9], for all nontrivial cycles γ of $\Gamma(A^{(1)})$, the equality in (4.6) becomes an equality. Let $u = (u_1, \ldots, u_n)^T$ be any eigenvector of $A^{(1)}$ corresponding to 0. By examining the known proofs (given in [6, Theorems 2.3 and 2.9] or [7, Theorems 3.6.4 and 3.6.9]) for the all-equalities case carefully, one can see that, in this case, for each $i \in \langle n \rangle$, if $u_i \neq 0$, then $|u_k|$ is nonzero and constant over all k for which $a_{ik}^{(1)} \neq 0$. By the strong connectedness of $\Gamma(A^{(1)})$, it follows that all u_j, $1 \leqslant j \leqslant n$, are nonzero. In addition, we also have $\mathfrak{M}(A^{(1)})|u| = 0$. From the definition of $A^{(1)}$ clearly we have $A(Du) = 0$. Furthermore, we have $\mathfrak{M}(A^{(1)}) = D^{-1}\mathfrak{M}(A)D$, and hence $\mathfrak{M}(A)|D| = 0$. By Theorem 2.1(b) $\Rightarrow$

(d), it follows that for all pairs $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, we have

$$\left| \sum_{l \neq i, j} a_{il} \bar{a}_{jl} - a_{ij} \bar{a}_{jj} - a_{ii} \bar{a}_{ji} \right| = \sum_{l=1}^{n} |a_{il}| |a_{jl}|.$$

This contradicts the second half of condition (e). ■

In [25, Theorem 4] Zhang and Gu showed that if λ is a complex number which is a boundary point of each of the $n(n-1)/2$ ovals of Cassini of an irreducible matrix A of order $\geqslant 3$, then necessarily λ is a boundary point of each of the n Geršgorins's disks of A. After they has obtained this result, they remarked that Brauer's theorem [4, Theorem 22] (that if λ is an eigenvalue of an irreducible matrix A that belongs to the boundary of the union of ovals of Cassini of A, then λ is a boundary point of each of the ovals of Cassini) is false except for trivial cases. By condition (b) of Theorem 4.4 we have found on the contrary that, if we rule out the case when the undirected graph of $\Gamma(A)$ is a star, then Brauer's theorem is always true, and in fact, in view of [25, Theorem 4], the stronger conclusion that the eigenvalue λ lies on the boundary of each of the n Geršgorin disks of A is valid.

5. EIGENVALUES ON THE BOUNDARY OF SPECTRUM INCLUSION REGIONS

In terms of location of eigenvalues Taussky's theorem [21, Theorem II] that every irreducibly diagonally dominant matrix is invertible is commonly reformulated in the following way:

Let A be an $n \times n$ irreducible matrix. A boundary point λ of the union of the Geršgorin disks

$$\{z : |z - a_{ii}| \leqslant R_i\}, \qquad i \in \langle n \rangle$$

is an eigenvalue of A only if λ is a boundary point of each of the disks.

We would like to point out that the above reformulation is actually slightly weaker than the original Taussky's theorem. To get the equivalent reformulation, we replace the geometric (but not readily checkable) assumption that λ

is a boundary point of the Geršgorin region of A by the weaker (but readily checkable) assumption that λ does not lie in the interior of each of the n Geršgorin disks. Here is an example to illustrate our point. Let

$$A = \begin{bmatrix} 1+i & \sqrt{2} & 0 & 0 \\ 0 & 1-i & \sqrt{2} & 0 \\ 0 & 0 & -1-i & \sqrt{2} \\ \sqrt{2} & 0 & 0 & -1+i \end{bmatrix}.$$

As can be readily verified, in this case we have $\det A = 0$; hence 0 is an eigenvalue of A. Also, it is clear that 0 lies on each Geršgorin circle. However, 0 is also an interior point of the Geršgorin region of A.

A similar remark also applies to other spectrum inclusion regions. (So, in a sense, the heading of this section is somewhat incomplete.)

Let A be an $n \times n$ irreducible matrix ($n \geqslant 2$), and let $\tau = (\tau_1, \dots, \tau_n)$ be an n-tuple of positive numbers such that $\mathfrak{M}^\tau(A)$ is an M-matrix. If λ is an eigenvalue of A for which we have

$$\prod_{P_i \in \gamma} |\lambda - a_{ii}| \geq \prod_{P_i \in \gamma} \tau_i$$

for all nontrivial cycles γ of $\Gamma(A)$, then what can we say about λ and its associated eigenvectors? In this section we are going to answer this and related questions.

First of all, since $\lambda I - A$ is singular, by Theorem 4.5, condition (d), we have

$$\prod_{P_i \in \gamma} |\lambda - a_{ii}| = \prod_{P_i \in \gamma} \tau_i$$

for all nontrivial cycles γ of $\Gamma(A)$. In this case, as mentioned in the proof of Theorem 4.5 [that condition (e) is sufficient for invertibility of A], there exists a diagonal matrix D with positive diagonal entries such that the matrix $A^{(1)} = D^{-1}(\lambda I - A)D$ has the property that each eigenvector of $A^{(1)}$ corresponding to 0 has nonzero components. It follow that if $x = (x_1, \dots, x_n)^T$ is any eigenvector of A corresponding to λ, then we have $x_i \neq 0$ for all $i \in \langle n \rangle$; hence, necessarily, the geometric multiplicity of λ equals 1. Furthermore, the vector x also satisfies $M(\lambda I - A)|x| = 0$. Hence, by

Theorem 2.1(b) $\Rightarrow$ (d), the following equalities hold:

$$|\lambda - a_{ii}|\,|a_{ji}| + |a_{ij}|\,|\lambda - a_{jj}| + \sum_{l \neq i,j} |a_{il}|\,|a_{jl}|$$

$$= \left| (\lambda - a_{ii})\bar{a}_{ji} + a_{ij}\,\overline{(\lambda - a_{jj})} + \sum_{l \neq i,j} a_{il}\bar{a}_{jl} \right|$$

for all pairs $(i,j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$.

In the case when $\tau = (R_1, \dots, R_n)$, something more can be said. As the known proofs show (see, for instance, the proof of Theorem 2.9 in [6]), then we have not only that $x_i \neq 0$ for all i, but also that for each i, $1 \leqslant i \leqslant n$, $|x_k|$ is constant over all k for which $P_k \in \Gamma^+(P_i)$. One consequence of this latter property is that from the (already proved) equation $M(\lambda I - A)|x| = 0$ it follows that for each $i \in \langle n \rangle$ we have

$$|\lambda - a_{ii}|\,|x_i| = R_i\,|x_k|, \tag{5.1}$$

where k is any index such that $P_k \in \Gamma^+(P_i)$. As can be readily shown, another consequence is that, if the (undirected) bipartite graph associated with the matrix $A - \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$ is connected, then we have

$$|x_1| = |x_2| = \cdots = |x_n|.$$

[For an $n \times n$ matrix A, we define its bipartite graph $B(A)$ as follows: The vertex set of $B(A)$ is the disjoint union of $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$, edges of $B(A)$ link only vertices of U with those of V, and there is an edge between u_i and v_j if and only if $a_{ij} \neq 0$.] But, in general, the $|x_i|$'s take different values.

Now, consider what happens when the eigenvalue λ satisfies the stronger condition that

$$|\lambda - a_{ii}|\,|\lambda - a_{jj}| \geqslant \tau_i \tau_j$$

for all pairs $(i,j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, for which P_i, P_j lie on a common cycle of $\Gamma(A)$. Again by Theorem 4.5 the inequalities in our condition are all equalities. If $\Gamma(A)$ has a nontrivial cycle of length $\geqslant 3$, then, arguing by way of contradiction and using the strong connectedness of $\Gamma(A)$ (cf. the proof of Lemma 4.3), we readily infer from our condition that $|\lambda - a_{ii}| = \tau_i$ for all $i \in \langle n \rangle$.

It remains to consider the case when $\Gamma(A)$ has no nontrivial cycle of length greater than 2. [Since A is irreducible, this amounts to saying that A is combinatorially symmetric and the undirected graph of $\Gamma(A)$ is a tree.] Suppose it is not the case that $|\lambda - a_{ii}| = \tau_i$ for all i; say, $|\lambda - a_{kk}| \neq \tau_k$. Take some $l \neq k$ such that P_k and P_l both lie on a cycle of length 2; clearly such an l exists. Then our condition implies that $|\lambda - a_{ll}|/\tau_l$ is simply the reciprocal of $|\lambda - a_{kk}|/\tau_k$. By the graph structure of $\Gamma(A)$ and by our condition, it follows that the n numbers $|\lambda - a_{ii}|/\tau_i$, $i \in \langle n \rangle$, take precisely two different values, one being the reciprocal of the other. Furthermore, for any pair $(i,j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, we have $|\lambda - a_{ii}|/\tau_i = |\lambda - a_{jj}|/\tau_j$ if and only if the distance from P_i to P_j (which is the same as the distance from P_j to P_i) in $\Gamma(A)$ is an even number. In other words, among the $n(n-1)/2$ ovals of Cassini

$$O_{ij} = \{z : |z - a_{ii}|\,|z - a_{jj}| \leqslant \tau_i\tau_j\}, \qquad 1 \leqslant i < j \leqslant n,$$

λ belongs to the boundary of precisely those O_{ij} for which the distance from P_i to P_j is an odd number.

When $\tau = (R_1, \dots, R_n)$, something more can be said about the components of an eigenvector x associated with λ. If $|\lambda - a_{ii}| = R_i$ for all $i \in \langle n \rangle$, then as is well known, we have $|x_1| = |x_2| = \cdots = |x_n|$ (see, for instance, [16, Theorem 6.2.8]). So, suppose that $|\lambda - a_{ii}| \neq R_i$ for some i, and hence for all i. [Then $\Gamma(A)$ has no cycles of length $\geqslant 3$.] In this case, as we have shown in (5.1), we have

$$|\lambda - a_{ii}|/R_i = |x_j|/|x_i|$$

whenever $a_{ij} \neq 0$ $(i \neq j)$. But we have also shown that the n numbers $|\lambda - a_{ii}|/R_i$, $i \in \langle n \rangle$, take precisely two different values. Using the graph structure of $\Gamma(A)$, we can then deduce that the n numbers $|x_i|$, $1 \leqslant i \leqslant n$, also take two different values and is in such a way that $|x_i| = |x_j|$ if and only if the distance from P_i to P_j is an even number.

Now suppose that λ satisfies the even stronger condition that

$$|\lambda - a_{ii}|\,|\lambda - a_{jj}| \geqslant \tau_i\tau_j$$

for all pairs $(i,j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$. Suppose, in addition, that for some (and hence, for all) i, $1 \leqslant i \leqslant n$, we have $|\lambda - a_{ii}| \neq \tau_i$. Then there can exist only one i, say $i = r$ such that $|\lambda - a_{rr}|/\tau_r < 1$. Also, $\Gamma(A)$ cannot have cycles of length $\geqslant 3$. Note that for any pair $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, $i \neq r \neq j$, there cannot exist arcs in $\Gamma(A)$ between P_i and P_j; otherwise, the strict inequality

$$|\lambda - a_{ii}||\lambda - a_{jj}|/\tau_i\tau_j > 1$$

would imply [by Theorem 4.5, condition (c)] the invertibility of $\lambda I - A$.

In order for $\Gamma(A)$ to be strongly connected, for any $i \neq r$, there must exist arcs from P_i to P_r and from P_r to P_i. Hence, the undirected graph of $\Gamma(A)$ is a star with center at P_r. Furthermore, among the $n(n-1)/2$ ovals of Cassini

$$O_{ij} = \left\{ z : |z - a_{ii}||z - a_{jj}| \leqslant \tau_i\tau_j \right\}, \qquad 1 \leqslant i < j \leqslant n,$$

λ belongs to the boundary of precisely $n - 1$ of them, namely, O_{ri}, for $i \in \langle n \rangle \setminus \{r\}$. If $\tau = (R_1, \dots, R_n)$, by the preceding discussion we also readily see that the components of the eigenvector x satisfy

$$|x_1| = |x_2| = \cdots = |x_{r-1}| = |x_{r+1}| = \cdots = |x_n| < |x_r|.$$

As the reader will see, our above discussion also explains and clarifies the results of [25]. (See, in particular, Theorem 3 and Corollary 2 of [25]. We also spotted a gap in the proof of case 2 of Lemma 3 (which is needed for Theorem 3), but we have found a different proof for the lemma.)

The following example shows that even when the eigenvalue λ satisfies

$$|\lambda - a_{ii}| = \tau_i \qquad \text{for all} \quad i \in \langle n \rangle,$$

λ need not be a simple eigenvalue.

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ \frac{1}{9} & 0 & -\frac{1}{3} & 0 \\ \frac{1}{9} & 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

Here we take $\tau = (R_1^{1/2}C_1^{1/2}, \dots, R_4^{1/2}C_4^{1/2})$. As can be readily checked, 0 is an eigenvalue of A that satisfies

$$|a_{ii}| = R_i^{1/2}C_i^{1/2} \qquad \text{for} \quad i = 1, \dots, 4.$$

Indeed, the geometric multiplicity of 0 as an eigenvalue is one, $x = (3, 1, 1, 1)^T$ being a corresponding eigenvector. On the other hand, its algebraic multiplicity is two (since the sum of all 3×3 principal minors of A equals zero).

6. REFINEMENTS OF SOLOV'EV'S THEOREM

In [9] Brualdi and Mellendorff, using a geometric argument, first proved an equivalent form of Theorem 3 of Pupkov [19] and then used it to derive a theorem of Solov'ev, which was stated as Theorem B in our introductory section. By modifying their arguments, we obtain the following refinements of Solov'ev's theorem.

THEOREM 6.1. *Let $A = (a_{ij})$ be an $n \times n$ irreducible matrix, $n \geqslant 2$. Let r be an integer with $1 \leqslant r \leqslant n$. Then A is invertible if it satisfies one of the following three sets of conditions:*

(I) (a) *For each $j \in \langle n \rangle$, either $|a_{jj}|$ is greater than the sum of the moduli of the $r - 1$ largest off-diagonal entries in the jth column, or $|a_{jj}|$ is equal to this sum and the modulus of the $(r - 1)$th largest off-diagonal entry in the jth column is nonzero; and*

(b) *either the sum of r smallest of the numbers $|a_{11}| - R_1, |a_{22}| - R_2, \dots, |a_{nn}| - R_n$ is positive, or this sum is equal to zero and the $n - r + 1$ largest of these n numbers are not all equal.*

(II) (a) *For each $j \in \langle n \rangle$, either $|a_{jj}|$ is greater than the sum of the moduli of the $r - 1$ largest off-diagonal entries in the jth column, or $|a_{jj}|$ is equal to this sum and the moduli of the smallest $n - r + 1$ off-diagonal entries in the jth column are not all equal; and*

(b) *either the sum of the r smallest of the numbers $|a_{11}| - R_1, |a_{22}| - R_2, \dots, |a_{nn}| - R_n$ is positive, or this sum is equal to zero and the rth smallest of these n numbers is positive.*

(III) *Conditions* I(a) *and* II(b) *are both satisfied and there exists a pair $(i, j) \in \langle n \rangle \times \langle n \rangle$, $i \neq j$, such that*

$$\sum_{l=1}^{n} |a_{li}|\,|a_{lj}| > \left| \sum_{l \neq i, j} a_{li}\bar{a}_{lj} - a_{ji}\bar{a}_{jj} - a_{ii}\bar{a}_{ij} \right|.$$

Proof. When $r = 1$ or when the smallest of the n numbers $|a_{11}| - R_1, \dots, |a_{nn}| - R_n$ is nonnegative, by condition (b) of (I) or (II), A is irreducibly diagonally dominant. When $r = n$, the sets of conditions (I), (II), and (III) each imply that A^T is irreducibly diagonally dominant. So in all these cases our theorem is valid.

Hereafter we assume that $1 < r < n$, and also that the smallest of the n numbers $|a_{11}| - R_1, \ldots, |a_{nn}| - R_n$ is negative. In this case, as can be readily seen, condition (I)(b) is stronger than (II)(b), whereas (II)(a) is stronger than (I)(a). To show that the set of conditions (I), (II), or (III) is sufficient for invertibility of A, we argue by way of contradiction and assume that there exists a nonzero vector $x = (x_1, \ldots, x_n)^T$ such that $x^T A = 0$. By normalizing the vector x, we may assume that for some index k we have $1 = |x_k| \geqslant |x_i|$ for all $i \in \langle n \rangle$. We are going to show that if conditions (I)(a) and (II)(b) are both satisfied, then x has nonzero components and, in addition, we have $|x|^T \mathfrak{M}(A) = 0$.

More or less, we are going to follow the argument given in [9, pp. 979–983] that leads to Theorem 5 there. By Lemmas 1.2 and 1.3 of [9], we have

$$\sum_{i \neq k} |a_{ik}|\, |x_k| \geqslant |a_{kk}| \tag{6.2}$$

and

$$\sum_{i \neq k} \left(|a_{ii}| - R_i\right) |x_i| \leqslant R_k - |a_{kk}|. \tag{6.3}$$

As in [9], we denote by Q_{n-1} the unit cube $\{(a_1, a_2, \ldots, a_{n-1})^T : 0 \leqslant a_i \leqslant 1$ for $1 \leqslant i \leqslant n-1\}$. Clearly the vector

$$u = \left(|x_1|, \ldots, |x_{k-1}|, |x_{k+1}|, \ldots, |x_n|\right)^T$$

belongs to Q_{n-1}. Denote by H the hyperplane of $\mathbb{R}^{n-1}$ consisting of vectors the sum of whose components equal $r - 1$. We contend that $u \in H$. The hyperplane H divides Q_{n-1} into two closed convex sets Q_{n-1}^- and Q_{n-1}^+, where Q_{n-1}^- (Q_{n-1}^+) consists of vectors in Q_n the sums of whose components are not greater than (not less than) $r - 1$. If $u \notin H$, then we have either $u \in Q_{n-1}^- \setminus H$ or $u \in Q_{n-1}^+ \setminus H$. We treat the case when $u \in Q_{n-1}^- \setminus H$ first. Then u can be expressed as

$$u = \lambda_1 v^1 + \cdots + \lambda_q v^q \tag{6.4}$$

where each $\lambda_j > 0$, $\Sigma_{j=1}^q \lambda_j = 1$, and $v^1, \dots, v^q$ are extreme points of Q_{n-1}^-. As an extreme point of Q_{n-1}^- each v^j is an $(n-1)$tuple of 0's and 1's with at most $r-1$ 1's. Furthermore, since we are assuming that $u \notin H$, at least one v^j has at most $r-2$ 1's. Let $v^j = (v_1^j, \dots, v_{k-1}^j, v_{k+1}^j, \dots, v_n^j)^T$ for each $j \in \langle q \rangle$. By condition (I)(a) clearly each v^j satisfies

$$\sum_{i \neq k} |a_{ik}| v_i^j \leqslant |a_{kk}|. \tag{6.5}$$

Condition (I)(a) also implies that the sum of moduli of the $r-2$ largest off-diagonal entries in the kth column is less than $|a_{kk}|$. Hence, for a v^j with at most $r-2$ 1's, (6.5) must hold with strict inequality. By (6.4) and the definition of u, it follows that we have

$$\sum_{i \neq k} |a_{ik}| \, |x_i| < |a_{kk}|,$$

which contradicts (6.2).

We have shown that $u \notin Q_{n-1}^- \setminus H$. Now consider $u \in Q_{n-1}^+ \setminus H$. In this case, the representation of u as a convex combination (with positive coefficients) of the extreme points $v^1, \dots, v^q$ (of Q_{n-1}^+) as given by (6.4) still holds, except that now each v^j is an $(n-1)$-tuple of 0's and 1's with at least $r-1$ 1's, and there is a v^j with at least r 1's. Clearly, by condition (II)(b), for any subset I of $\{1, 2, \dots, n\} \setminus \{k\}$ with $r-1$ or more elements, we have

$$(|a_{kk}| - R_k) + \sum_{i \in I} (|a_{ii}| - R_i) \geqslant 0. \tag{6.6}$$

By condition (II)(b) the rth smallest of the n numbers $|a_{11}| - R_1, \dots, |a_{nn}| - R_n$ is positive. Hence, (6.6) holds with strict inequality if I has more than $r-1$ elements. It follows that each extreme point v^j that appears in the representation (6.4) for u must satisfy

$$\sum_{i \neq k} (|a_{ii}| - R_i) v_i^j \geqslant R_k - |a_{kk}|, \tag{6.7}$$

with strict inequality if v^j has at least r 1's. Thus, we conclude that

$$\sum_{i \neq k} (|a_{ii}| - R_i) \, |x_i| > R_k - |a_{kk}|,$$

which contradicts (6.3). This shows that $u \notin Q_{n-1}^+ \setminus H$.

The preceding argument has established the contention that $u \in H$. Indeed, it also shows that there is a convex representation of u given by (6.4), in which each v^j is an $(n-1)$-tuple of 0's and 1's with exactly $r-1$ 1's and is such that (6.7) is satisfied as an equality; hence, the equality holds in (6.3), and moreover the sum of the r smallest numbers among $|a_{ii}| - R_i$, $1 \leqslant i \leqslant n$, is equal to 0, and $|a_{kk}| - R_k$ is among one of these r numbers.

From the proof of Lemma 1.3 in [9] one readily sees that when the equality holds in (6.3), necessarily we have

$$\sum_{i \neq j} |a_{ij}|\,|x_i| = |a_{jj}|\,|x_j| \qquad \text{for } j = 1, 2, \ldots, n.$$

In other words, the vector $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ satisfies the equation $|x|^T \mathfrak{M}(A) = 0$. [Indeed, the equality holds in (6.3) if and only if $|x|^T \mathfrak{M}(A) = 0$.] But $\mathfrak{M}(A)$ is an irreducible Z-matrix; it follows $|x|$ is a nonnegative eigenvector of some irreducible nonnegative matrix and hence is a positive vector. Thus the vector x has nonzero components. This proves our initial claim concerned with a nonzero vector x that satisfies $x^T A = 0$, provided that conditions (I)(a) and (II)(b) are both satisfied.

Now suppose that the set of conditions (I) is fulfilled. As we have proved, $|a_{kk}| - R_k$ is among the r smallest numbers of $|a_{ii}| - R_i$, $1 \leqslant i \leqslant n$. Let $|a_{pp}| - R_p$ be the greatest of the n numbers $|a_{ii}| - R_i$, $1 \leqslant i \leqslant n$. In view of condition (I)(b), clearly $p \neq k$. Also, the sum of the numbers $|a_{pp}| - R_p$, $|a_{kk}| - R_k$, and any other $r-2$ numbers taken from $|a_{ii}| - R_i$, $1 \leqslant i \leqslant n$, is greater than zero. It follows that the pth component of each vector v^j that appears in the representation (6.4) for the vector u is zero; hence $|x_p|$ (the pth component of u) is also zero. This contradicts the proved fact that x has nonzero components.

When the set of conditions (II) is fulfilled, we can use condition (II)(a) and the fact that the equality holds in (6.2) [which is the kth equation of $|x|^T \mathfrak{M}(A) = 0$] to draw the conclusion that x has a zero component, and hence a contradiction.

When the set of conditions (III) is fulfilled, by conditions (I)(a) and (II)(b), any nonzero vector x that satisfies $x^T A = 0$ must have nonzero components; in addition, we have $|x|^T \mathfrak{M}(A) = 0$. Hence by Theorem 2.1(b) $\Rightarrow$ (d), condition (d)(ii) of Theorem 2.1 must be satisfied by the matrix A^T. This contradicts the second half of condition (III).

In the statement of Theorem 6.1, if we require the smallest of the n numbers $|a_{11}| - R_1, \ldots, |a_{nn}| - R_n$ to be negative (i.e. if we rule out the case when A is diagonally dominant, which is already covered by known results), then we can replace condition (II)(b) simply by the following: The sum of the r smallest of the numbers $|a_{11}| - R_1, \dots, |a_{nn}| - R_n$ is nonnegative.

The following example shows that conditions I(a) and II(b) of Theorem 6.1 together are not sufficient for an irreducible matrix A to be invertible.

EXAMPLE 6.2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

Take $r = 2$. As can be readily checked, A satisfies both of the conditions I(a) and II(b) of Theorem 6.1, but not the conditions II(a) (for $j = 1$) and I(b). Note that the matrix A here can be obtained from the one that appears at the end of Section 5 by premultiplying it by $\mathrm{diag}(1, -9, 9, 9)$ and postmultiplying it by $\mathrm{diag}(1, -1, -1, -1)$. But the latter matrix is singular; hence so is A.

As can be readily seen, Theorem 5 of [20] also follows as a corollary of our Theorem 6.1.

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